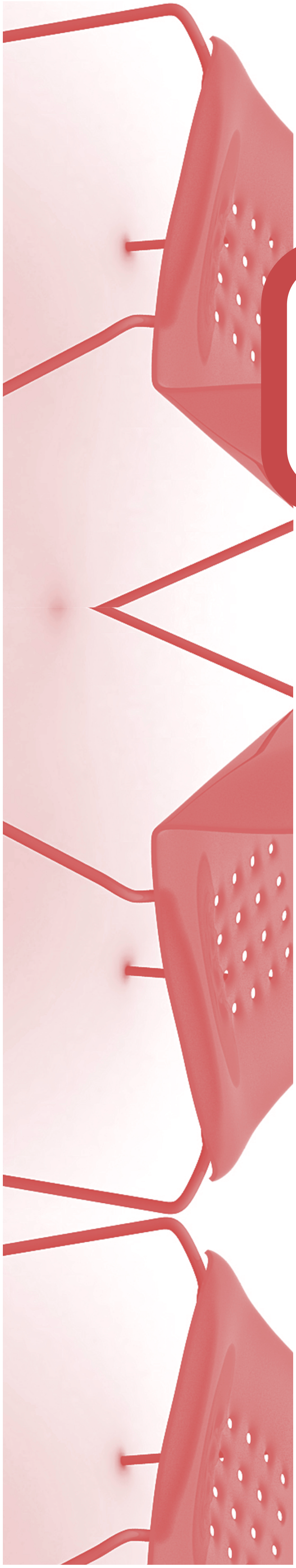


26

GUIDE TO CALCULUS

We have seen in Chapter 18 how a range of business situations and economic relationships can be described mathematically. Using equations, we can determine the breakeven point of a company or the price where market demand is equal to market supply. However, we are not always just interested in a particular level of output, or price, or point in time but often how *change* is taking place. We may be interested in the rate at which profits are increasing or decreasing, or the effects of a unit change in price, or sales growth over time.

Calculus is about the measurement of change. It provides the methodology to calculate the rate of change and indeed, whether the rate of change is increasing or decreasing. Calculus also provides a notation to describe change. Given that y is a function of x ($y = f(x)$), the change in y resulting from a change in x is written $\mathrm{d}y/\mathrm{d}x$.

Objectives

After working through this chapter, you should be able to:

- define differentiation
- differentiate simple functions of one variable
- apply this differentiation to economic models
- find maxima and minima of one-variable functions
- differentiate more complex one-variable functions
- differentiate functions of more than one variable
- find maxima and minima subject to constraints
- integrate simple functions.

26.1 Differentiation

In this section we consider an intuitive development of differentiation (26.1.1), and the rules of differentiation (26.1.2).

26.1.1 Differentiation: an intuitive approach

Differentiation measures the change in y resulting from a change in x. If we were looking at a graph plot of y against x, we would think of this change in terms of gradient; and this is helpful to some extent. Suppose we were interested in the cost of car hire. The total cost in pounds, y, could be plotted against the number of miles, x. The gradient would give the cost of each extra mile.

If the total cost was a fixed charge of £200, then $y = 200$. No x term is included as the cost does not depend on the number of miles travelled. The change in y resulting from a change in x is 0, as an extra mile makes no difference to the cost. In this case, $dy/dx = 0$:

If the cost of car hire had two components, a fixed charge of £150 and a mileage charge of 20p per mile, then $y = 150 + 0.20x$. Each extra mile would cost 0.20 and $dy/dx = 0.20$. In the case of a linear function, the gradient is constant (the increase in y resulting from a unit increase in x) and the change measured by the differential dy/dx is equal to the gradient.

The cost of car hire could be expressed in a more complex form:

$$\begin{aligned} y &= 150 + 0.30x - 0.0005x^2 && \text{for } 0 \le x \le 500 \\ y &= 150 + 0.04x && \text{for } x > 500 \end{aligned}$$

There are two major issues to consider here: continuity and the meaning of change. First, for differentiation to be valid the function must be *continuous* within the relevant range, i.e. no gaps or jumps. In this case, we are dealing with two functions, both continuous within the ranges given; differentiation is valid except at the boundary points of $x = 0$ and $x = 500$ where the functions are discontinuous. As we will see from the rules of differentiation given in section 26.1.2:

$$\begin{aligned} dy/dx &= 0.30 - 0.001x && \text{for } 0 < x < 500 \\ dy/dx &= 0.04 && \text{for } x > 500 \end{aligned}$$

The rate of change is not constant and will depend on the value of x

$$\begin{aligned} &\text{e.g. when } x = 100, && dy/dx = 0.2, \text{ and} \\ &\text{when } x = 200, && dy/dx = 0.1 \end{aligned}$$

But does dy/dx measure gradient? Now the y values corresponding to $x = 100$ and $x = 200$ are 175 ($150 + 0.30 \times 100 - 0.0005 \times 100^2$) and 190 ($150 + 0.03 \times 200 - 0.0005 \times 200^2$) respectively. This increase of 15 units in y achieved over an increase of 100 units in x gives an average increase of 0.15 units in y for each unit increase in x, over this range. Differentiation does not give the average increase over a range of x values but rather a measure of change occurring at a point on the curve. Differentiation can be thought of as giving the gradient of a tangent to a curve.

Before moving on to consider the rules of differentiation, let us again consider the equation $y = 150 + 0.30x - 0.0005x^2$. We are often presented with such equations and given little explanation, but it is worth pausing to examine the structure of this equation. It can be rewritten $y = 150 + (0.30 - 0.0005x)x$. The three components of cost are therefore a fixed cost, plus a charge per mile times the number of miles. The charge per mile, $(0.30 - 0.0005x)$, depends on the number of miles, x, and as x increases, the charge made decreases. You will also find this form of equation when you look at revenue functions where price per unit falls as sales increase. It is always worth asking the question 'What does this equation mean?'.

26.1.2 Differentiation: the rules

To obtain a value for dy/dx we can apply a number of rules:

1 Given a function of the form $y = k$ where k is a constant,

$$\frac{dy}{dx} = 0$$

A constant, by its nature, does not change (depend on x) and therefore has zero rate of change.

EXAMPLE

If $y = 23$, then $dy/dx = 0$.

2 Given a function of the form $y = ax$ where a is a constant,

$$\frac{dy}{dx} = a$$

For each unit increase in x there is a constant a unit increase in y – the rate of change.

EXAMPLE

If $y = 4.5x$, then $dy/dx = 4.5$

3 Given a function of the form $y = ax + k$

$$\frac{dy}{dx} = a$$

When we differentiate this more general linear function, the rate of change is still a constant given by the x coefficient, because for a function consisting of several parts or terms, we differentiate each part separately and then put the results together.

EXAMPLE

If $y = 32.8 + 0.96x$, then $dy/dx = 0.96$.

4 Given a function of the form $y = ax^n$

$$\frac{dy}{dx} = nax^{n-1}$$

EXAMPLE

If $y = 6x^3$, then $a = 6$ and $n = 3$

$$\frac{dy}{dx} = 3 \times 6x^2 = 18x^2$$

EXAMPLE

If $y = 10x^2$, then $dy/dx = 2 \times 10x^1 = 20x$ since $x^1 = x$.

EXAMPLE

If $y = 6x^3 - 4x^2 + 10x - 50 + 3x^{-2}$, then

$$\frac{dy}{dx} = 3 \times 6x^2 - 2 \times 4x^1 + 10x^0 + 0 + (-2) \times 3x^{-3}$$

$$= 18x^2 - 8x + 10 - 6x^{-3}$$

NB. $x^0 = 1$.

To complete this section, we need to consider certain other functions which have special *derivatives*. A derivative is the result of differentiation and is also called the differential coefficient.

EXAMPLE

If $y = e^x$ (the exponential function described in section 18.4.5) then $dy/dx = e^x$.

EXAMPLE

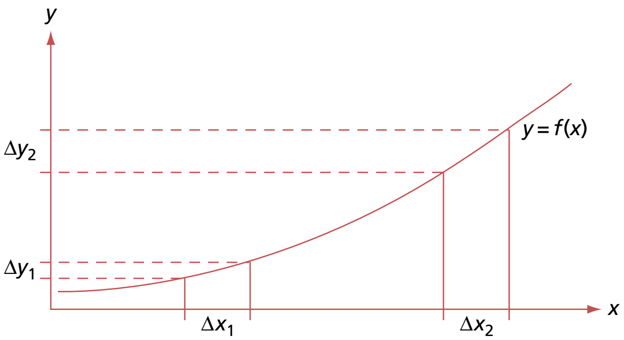
If $y = \log_e x$ then $dy/dx = 1/x$.

26.1.3 Differentiation: the theoretical development

Calculus, as discussed, is about the measurement of change. However, the rate of change can be interpreted in two ways: the average rate of change and the instantaneous rate of change. If you consider a car travelling from Birmingham to Manchester, then its **average speed** (i.e. the rate of change of distance in relation to time) may be 50 miles per hour, but its speed at any particular moment may be much less (for example 5 m.p.h. in heavy traffic) or rather more (for example 70 m.p.h. on the motorway). The driver, looking at the speedometer at a particular moment, can tell the **instantaneous speed**. Now it has been argued that there can be no such thing as instantaneous speed, since speed is to do with the distance travelled, but try convincing someone who has been driving and met a wall (which was presumably stationary) that they could not be travelling at a speed at the instant the vehicle and the wall met!

Calculus, then, is about the **instantaneous rate of change** and *not* the average rate of change.

Figure 26.1



If we consider the graph of a function (as in Figure 26.1) we can see that there is an increasing change in y ($\Delta y_1 < \Delta y_2$) for equal changes in x ($\Delta x_1 = \Delta x_2$) as we move to the right. The symbol Δ (Greek capital delta) indicates an interval in the value of a variable, e.g. Δx represents the difference between two values of x. Thus there are different rates of change in y for different values of x. To find the average rate of change between two values of x, we would divide the change in y by the change in x. Here,

$$\frac{\Delta y_1}{\Delta x_1} \text{ is less than } \frac{\Delta y_2}{\Delta x_2}$$

If a small section of the graph of the function is magnified (as in Figure 26.2), we see that the ratio $\Delta y/\Delta x$ not only measures the average rate of change of y between two values of x, but also the gradient or slope of a straight line (known as a **chord**) drawn between two points on the graph. As we reduce the change in x the straight line gets shorter and shorter and hence its slope or rate of change gets nearer and nearer to the slope of the curve at a single point; but if we have no change in x, then there is no change in y, although the function is still changing (cf. the car mentioned above). Now if the straight line between the two points can represent the average slope of the graph between the points, is there a straight line at one point whose slope is that of the graph at that point? The answer is yes!

Figure 26.2

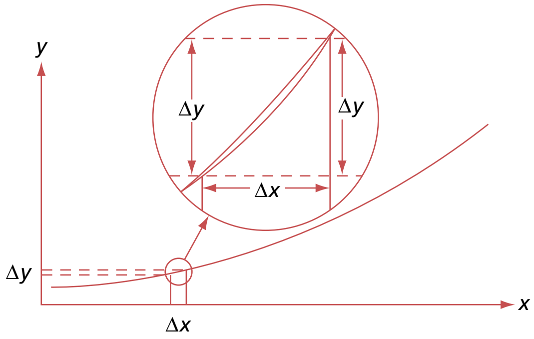
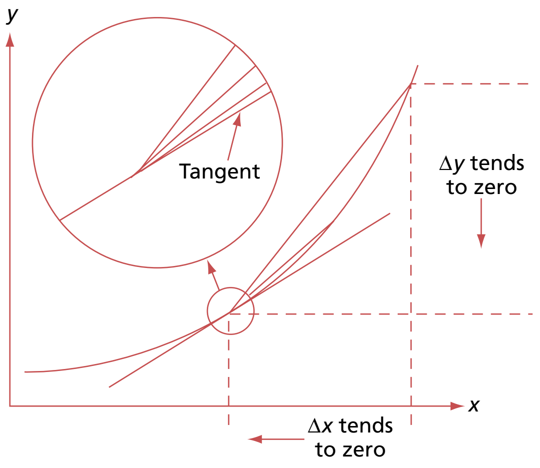


Figure 26.3 shows a straight line which touches the curve at just one point (known as a **tangent**) and the slope of this line will be the slope of the curve at the point where the two touch. Since there is only one point now in question, the slope of the curve at that point measures not the average but the instantaneous rate of change.

Figure 26.3



Formally, the measurement of the slope of a curve at a point is valid if the function is **continuous**, i.e. there are no gaps or jumps in the function; thus if $y = f(x)$ then both x and y are continuous variables in the relevant range or domain. The slope of the tangent at the point where it touches the function is defined as the limit of the ratio $\Delta y/\Delta x$ as Δx tends to zero.

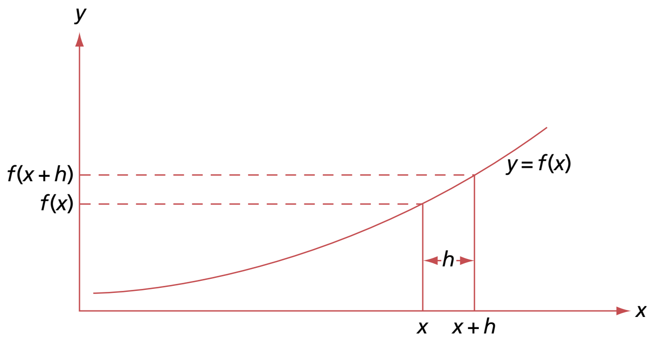
A **limit** is a number to which the ratio gets nearer and nearer, as the **interval** Δx gets smaller. For example the ratio

$$\frac{x + \Delta x}{x}$$

will get closer and closer to 1 as the interval Δx gets smaller. Thus the limit of this expression will be 1.

Formally, the change in x is thought of as a distance, h, so that the interval Δx is from x to $x + h$. Since $y = f(x)$, the interval Δy extends from $f(x)$ to $f(x + h)$ (see Figure 26.4).

Figure 26.4



We saw above that an average slope of a function was

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

or, in Figure 26.5

$$\frac{f(x + h) - f(x)}{(x + h) - x} = \frac{f\,(x + h) - f(x)}{h}$$

and it is this ratio, as h gets smaller and smaller, that gives the instantaneous rate of change of $f(x)$ at a single value of x.

The limit of $\Delta y/\Delta x$ as h :0 is denoted by $\mathrm{d}y/\mathrm{d}x$ or $f'(x)$. Here d signifies an infinitesimal change in a variable (cf. Δ which signifies finite change).

EXAMPLE

If $f(x) = x^2$, then $f(x + h) = (x + h)^2 = x^2 + 2xh + h^2$ so the ratio $\Delta y/\Delta x$ is given by:

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{(x^2 + 2xh + h^2) - x^2}{h}\\ &= \frac{2xh + h^2}{h}\\ &= 2x + h\end{aligned}$$

and as h : 0, this gets closer and closer to $2x$. Thus, for $y = x^2$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$$

So the rate of change of the function $y = x^2$ at $x = 2$, for example, is $2x = 2 \times 2 = 4$.

Note that the rate of change of a function is often referred to as the **gradient** of the function.

26.2 Economic applications I

Within economics, there are several functions which are related to each other as function to derivative. For instance, the total cost (TC) represents the cost of producing a particular amount of the product; the marginal cost (MC) is the cost of producing one extra unit and so the marginal cost of a given output is the rate at which total cost is changing at that output. Thus, if we have a total cost function and differentiate it, we will find that the result is a marginal cost function. This relationship will also hold for revenue functions.

Thus if $y = \mathrm{TC}$ then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{MC}$$

and if y = TR then

$$\frac{dy}{dx} = \text{MR}$$

where TR is total revenue and MR is marginal revenue. For example, if

$$\text{TC} = 40 + 10x + 2x^2 + x^3$$

then

$$\text{MC} = 10 + 4x + 3x^2$$

(per unit change in output when output = x). For example, if

$$\text{TR} = 4x$$

then

$$\text{MR} = 4$$

(per unit change in sales when sales = x).

We may also use the idea of averaging to find the average cost of, or revenue from, a given output. If total revenue is £100 from an output of 5, then the average revenue (AR) will be 100/5 = £20 per unit. So if

$$\begin{aligned}\text{TR} &= 100x - 10x^2\\ \text{AR} &= \frac{1}{x}(100x - 10x^2)\\ &= 100 - 10x\end{aligned}$$

(per unit when x units are sold). Similarly if

$$\text{AC} = x^2 - 10x + 38$$

then

$$\begin{aligned}\text{TC} &= x(x^2 - 10x + 38)\\ &= x^3 - 10x^2 - 38x.\end{aligned}$$

If we begin with a cubic total cost function and find the marginal cost function, which will be quadratic, and the average cost function, which will also be quadratic, then graph these, we will have the typical economic diagram, as in Figure 26.5.

With a linear demand curve (i.e. an AR function) we will obtain the relationship in Figure 26.6, where the marginal revenue function will also be linear, but have a slope twice that of the average revenue function. Since if

$$\text{AR} = a + bx$$

then

$$\text{TR} = ax + bx^2$$

Figure 26.5

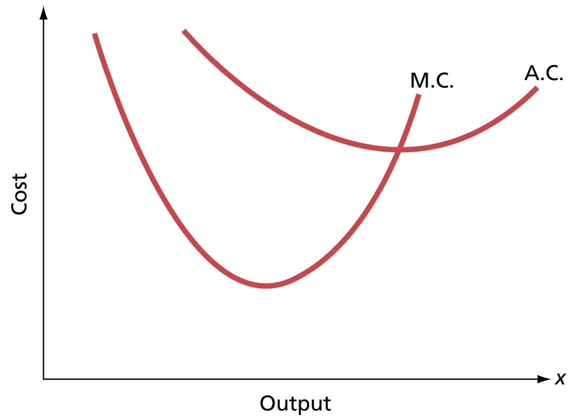
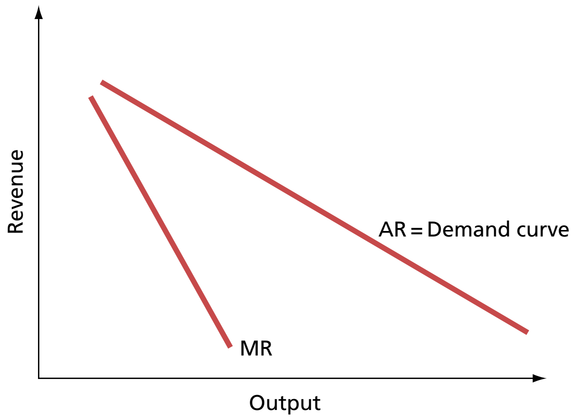


Figure 26.6



and

$$\mathrm{MR} = a + 2bx$$

Elasticity of demand measures the responsiveness of the quantity sold to various factors: initially, the most useful of these is price elasticity (E_D), which is defined as:

$$E_D = \frac{dq}{dp} \times \frac{p}{q}$$

EXAMPLE

If the demand curve is given by

$$p = 100 - 5q$$

then

$$\frac{dp}{dq} = -5$$

and since

$$\frac{dq}{dp} = \frac{1}{dp/dq}$$

then

$$\frac{dq}{dp} = -\frac{1}{5}$$

At a quantity of 10, the price is:

$$p = 100 - 5(10) = 50$$

So, elasticity is:

$$E_D = \frac{1}{-5} \times \frac{50}{10} = -1$$

At a quantity of 8, the price is 60 and elasticity is

$$E_D = \frac{1}{-5} \times \frac{60}{8} = -1.5$$

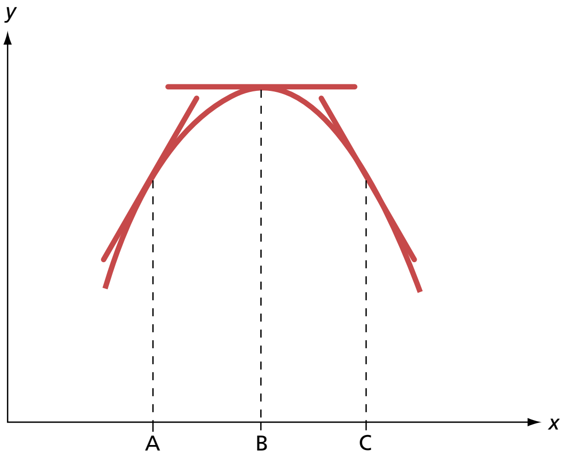
So for a linear demand function, there is a *constant* slope but a *changing* elasticity.

26.3 Turning points

In Chapter 18 we considered quadratic functions and stated that the sign of the coefficient of x^2 determined the shape of the function. With functions that include higher powers of x the graph will have several changes of direction. We can use the method of differentiation developed above to locate both *where* these changes of direction occur and which way the change affects the function. This is the method of locating the *maximum* and *minimum* values of a function. By a 'maximum' value we do not necessarily mean one that is greater than all other values (a *global* maximum), it could simply be greater than all neighbouring values only (a *local* maximum). Similar definitions apply to minimum values. For the economic applications at this stage, this distinction between local and global maxima is not of great importance.

If you consider the function represented in Figure 26.7 you can see that the slopes of the function at points A, B and C are quite different. At point A the function is increasing and

Figure 26.7



thus the slope is positive. At point *C* the function is decreasing, and thus the slope is negative. However, at point *B*, it is just changing direction and is neither going up nor going down; therefore the slope is zero. Now the slope of a function can be found by differentiation.

In Figure 26.8, the first graph represents a function which has a maximum at the point *A* and a minimum at the point *B*, and the second graph shows dy/dx against x for this function. This shows that to the left of the point *A* there is a positive slope, but that slope is decreasing. At *A* there is zero slope, and so $dy/dx = 0$. After *A* the original function is decreasing and so dy/dx is negative, but when the point *B* is reached, where the original function begins to go up again, there will be zero slope, and dy/dx will again be 0.

To the right of point *B*, the original function is increasing, and hence the second graph is in the positive region.

From this we see that at both *turning points* the value of $dy/dx = 0$:

There is more information that we can gain from Figure 26.8. Looking at the second graph we find that at the point *A*, the value of dy/dx is decreasing as x increases, and thus has a negative slope, while at the point *B*, the value of dy/dx is increasing, and thus has a positive slope. Thus, if we can find the slope of the dy/dx function, we can distinguish between maximum points like *A* and minimum points like *B*.

In effect, this means differentiating the original function a second time, and we denote this second differential by a slightly different symbol:

$$\frac{d^2y}{dx^2}$$

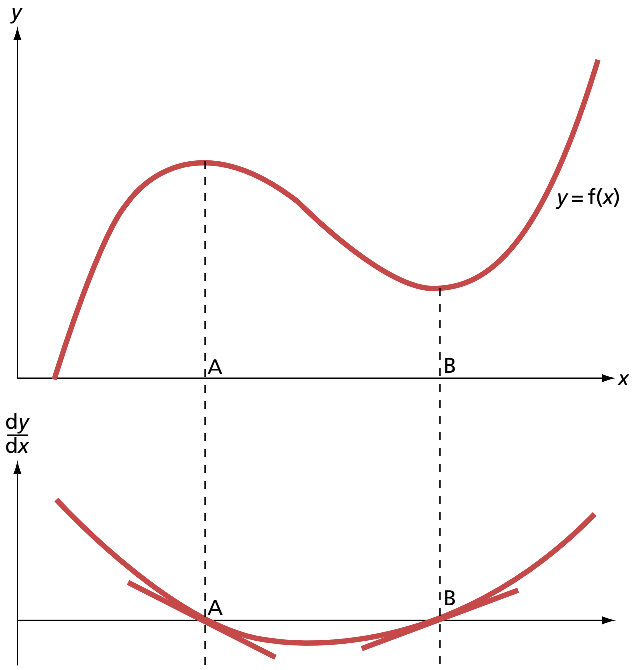
The process of differentiating is still the same, only it is now applied to the function dy/dx.

The rules for distinguishing a maximum from a minimum value are thus:

$$\text{for a maximum, } \frac{dy}{dx} = 0 \text{ and } \frac{d^2y}{dx^2} < 0$$

$$\text{for a minimum, } \frac{dy}{dx} = 0 \text{ and } \frac{d^2y}{dx^2} > 0$$

Figure 26.8



EXAMPLE

If

$$y = 2x^2 - 8x + 50$$

then

$$\frac{dy}{dx} = 4x - 8 = 0$$

therefore $4x = 8$ and hence

$$x = \frac{8}{4} = +2$$

so the turning point is at $x = 2$.

$$\frac{d^2y}{dx^2} = 4$$

which is *positive*, and so the turning point is a minimum.

The value of y is $2(2)^2 - 8(2) + 50 = 8 - 16 + 50 = 42$.
Thus the function $y = 2x^2 - 8x + 5$ has a minimum point at (2, 42).

EXAMPLE

$$y = \tfrac{1}{3}x^3 - 4x^2 + 15x + 10$$

then

$$\frac{dy}{dx} = x^2 - 8x + 15 = 0$$

factorising gives $(x - 3)(x - 5) = 0$ so $x = 3$ and 5 and there is a turning point for each of these values.

$$\frac{d^2y}{dx^2} = 2x - 8$$

Now at $x = 3$, $2x - 8 = 2(3) - 8 = -2$ (negative $\Rightarrow$ maximum) and at $x = 5$, $2x - 8 = 2(5) - 8 = +2$ positive $\Rightarrow$ minimum).
Also, at $x = 3$,

$$y = \tfrac{1}{3}(3)^3 - 4(3)^2 + 15(3) + 10 = 28$$

and, at $x = 5$,

$$y = \tfrac{1}{3}(5)^3 - 4(5)^2 + 15(5) + 10 = 26.67$$

Thus the function

$$y = \tfrac{1}{3}x^3 - 4x^2 + 15x + 10$$

has a maximum at (3, 28) and a minimum at (5, 26.67).

26.4 Economic applications II

The economist is often interested in finding the maximum value of certain relationships for profit, sales revenue, welfare, etc., and minimum values, particularly for cost functions. The methods developed above will allow us to specify where these turning points occur.

EXAMPLE

If a firm has

$$TR = 40x - 8x^2$$

and

$$TC = 8 + 16x - x^2$$

where x = thousands of units of product, then its profit function (π) will be the difference between its total revenue and total cost:

$$\begin{aligned}\pi &= TR - TC\\ &= (40x - 8x^2) - (8 + 16x - x^2)\\ &= 40x - 8x^2 - 8 - 16x + x^2\\ &= -8 + 24x - 7x^2\end{aligned}$$

If we wish to find where maximum profit occurs, we differentiate to give

$$\frac{d\pi}{dx} = 24 - 14x$$

and find that $d\pi/dx$ is zero when $x = 24/14 = 1.714$, and $d^2\pi/dx^2 = -14$, which is negative, thus representing a maximum.

So the firm will achieve maximum profit at an output of $x = 1.714$, and since x is measured in thousands, this is 1714 units. Profit here is 12.571. If the firm wishes to maximize its sales revenue (i.e. TR), we have:

$$TR = 40x - 8x^2$$

$$\frac{dTR}{dx} = 40 - 16x$$

and when $dTR/dx = 0$, $x = 40/16 = 2.5$ and $\dfrac{d^2TR}{dx^2} = -16$, which is negative, thus representing a maximum.

So, maximum sales revenue will be at 2500 units, but profit here would only be 8.25.

EXAMPLE

If a firm's total cost function is

$$TC = 200 + 10x - 6x^2 + x^3$$

then the 200 represents fixed cost, since it does not vary with the level of output. If we remove this fixed cost, we will be left with total variable cost (TVC) with

$$TVC = 10x - 6x^2 + x^3$$

and average variable cost (AVC) can be found by dividing by x:

$$\text{AVC} = 10 - 6x + x^2$$

We want to know the output to give minimum AVC, and to find this, we differentiate:

$$\frac{\text{dAVC}}{\text{d}x} = -6 + 2x = 0$$

therefore $x = 3$ and

$$\frac{\text{d}^2\text{AVC}}{\text{d}x^2} = +2$$

positive, thus minimum.

So, minimum average variable cost is at an output of 3.

26.5 Further notes

1 If the function to be differentiated is the product of two functions, then there is a method of differentiating without having to multiply the two functions out, e.g.

$$y = (2x^2 + 10x + 5)(6x^3 + 12x^2)$$

let

$$u = 2x^2 + 10x + 5$$

and

$$v = 6x^3 + 12x^2$$

so if $y = uv$ then, in general,

$$\frac{\text{d}y}{\text{d}x} = v \times \frac{\text{d}u}{\text{d}x} + u \times \frac{\text{d}v}{\text{d}x}$$

Now if $u = 2x^2 + 10x + 5$ then

$$\frac{\text{d}u}{\text{d}x} = 4x + 10$$

and if $v = 6x^3 + 12x^2$ then

$$\frac{\text{d}v}{\text{d}x} = 18x^2 + 24x$$

and

$$\begin{aligned}\frac{\text{d}y}{\text{d}x} &= (6x^3 + 12x^2)(4x + 10) + (2x^2 + 10x + 5)(18x^2 + 24x)\\ &= 24x^4 + 60x^3 + 48x^3 + 120x^2 + 36x^4 + 48x^3 + 180x^3\\ &\quad + 240x^2 + 90x^2 + 120x\\ &= 60x^4 + 336x^3 + 450x^2 + 120x\end{aligned}$$

2 If the function to be differentiated consists of one function divided by another, then the following method is appropriate.

$$y = \frac{(10x^2 + 6x + 5)}{(12x^3 + 15x^2)}$$

so let

$$u = 10x^2 + 6x + 5$$

and

$$v = 12x^3 + 15x^2$$

Then if $y = u/v$, then, in general,

$$\frac{dy}{dx} = \frac{v \times \dfrac{du}{dx} - u \times \dfrac{dv}{dx}}{v^2}$$

If $u = 10x^2 + 6x + 5$ then

$$\frac{du}{dx} = 20x + 6$$

and if $v = 12x^3 + 15x^2$ then

$$\frac{dv}{dx} = 36x^2 + 30x$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(12x^3 + 15x^2)(20x + 6) - (10x^2 + 6x + 5)(36x^2 + 30x)}{(12x^3 + 15x^2)^2} \\
&= \frac{\begin{array}{l}240x^4 + 300x^3 + 72x^3 + 90x^2 \\ -(360x^4 + 300x^3 + 216x^3 + 180x^2 + 180x^2 + 150x)\end{array}}{144x^6 + 360x^5 + 225x^4} \\
&= \frac{-120x^4 - 144x^3 - 270x^2 - 150x}{144x^6 + 360x^5 + 225x^4} \\
&= \frac{-6x(20x^3 + 24x^2 + 45x + 25)}{x^4(144x^2 + 360x + 225)}
\end{aligned}$$

Note that it is not always necessary to carry out all of the multiplications and simplifications can help if made earlier in the calculation.

3 If the function to be differentiated is a function of another function, then the following method is appropriate. If

$$y = (10x^2 + 6x)^3$$

let $u = 10x^2 + 6x$ thus

$$\frac{du}{dx} = 20x + 6$$

and $y = u^3$ thus

$$\frac{dy}{du} = 3u^2$$

and, in general,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 3u^2(20x + 6) \\ &= 3(10x^2 + 6x)^2\,(20x + 6) \\ &= 3(100x^4 + 120x^3 + 36x^2)(20x + 6)\end{aligned}$$

This could be further simplified.

4 If the function is exponential then if $y = e^x$

$$\frac{dy}{dx} = e^x$$

If $y = e^{ax}$ then

$$\frac{dy}{dx} = ae^{ax}$$

e.g.

$$y = e^{2x} \text{ and } \frac{dy}{dx} = 2e^{2x}$$

If $y = e^{(x^2+6)}$ let $u = x^2 + 6$. Then

$$y = e^u \text{ and } \frac{du}{dx} = 2x$$

so

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = e^u \times 2x = 2xe^{(x^2+6)}$$

5 If $y = \log_e x$ then

$$\frac{dy}{dx} = \frac{1}{x}$$

If

$$y = \log_e(x^3 + 2x + 3)$$

let

$$u = x^3 + 2x + 3$$

then

$$y = \log_e(u) \quad \text{and} \quad \frac{du}{dx} = 3x^2 + 2$$

so

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \frac{1}{u}(3x^2 + 2) \\ &= \frac{3x^2 + 2}{x^3 + 2x + 3}\end{aligned}$$

26.6 Integration

Differentiation has allowed us to find an expression for the rate of change of a function; what we need now is some method of reversing the process, i.e. obtaining the original function when the rate of change is known. This process is *integration*.

If $y = x^2$, we know that the rate of change is $2x$; thus if we integrate $2x$ we must get x^2. However, if $y = x^2 + 10$, the rate of change is still $2x$, and integrating will give us x^2 and *not* $x^2 + 10$. This is because the constant in the initial expression has a zero rate of change, and therefore disappears when it is differentiated. When we integrate, then, we should add a constant c to the expression we obtain, and we will need some further information if we are to find the specific value of this constant. For example, if

$$\frac{dy}{dx} = 2x$$

integrating gives

$$y = x^2 + c$$

If we also know that $y = 10$ when $x = 0$ we have

$$10 = 0^2 + c$$

therefore $c = 10$ and hence $y = x^2 + 10$:

The symbol used for integration is an ‘old style’ S, i.e. ∫, and it is usual to put dx after the expression to show that we are integrating with respect to x. Thus

$$\int (2x)\, dx = x^2 + c$$

As with differentiation, there is a general formula for integration. If

$$y = ax^n$$

where a is a constant then

$$\int y\,dx = \frac{ax^{n+1}}{n+1} + c$$

for all values of n except $n = -1$. In that special case we have

$$\int \frac{1}{x}\,\mathrm{d}x = \log_e x + c$$

Reverting to the more usual case, if $y = 15x^4$, then $a = 15$ and $n = 4$ so

$$\begin{aligned}\int y\mathrm{d}x &= \frac{15x^{4+1}}{4+1} + c = \frac{15x^5}{5} + c\\ &= 3x^5 + c\end{aligned}$$

When there are several terms in the function we must treat each separately, e.g.

$$\begin{array}{llll} y = 10x^3 & +\,6x^2 & -4x & +\,10\\ \quad(a = 10 & (a = 6 & (a = -4 & (a = 10\\ \quad\ n = 3) & \ n = 2) & \ n = 1) & \ n = 0)\end{array}$$

$$\begin{aligned}\int y\mathrm{d}x &= \frac{10x^{3+1}}{3+1} + \frac{6x^{2+1}}{2+1} - \frac{4x^{1+1}}{1+1} + 10x + c\\ &= \frac{10x^4}{4} + \frac{6x^3}{3} - \frac{4x^2}{2} + 10x + c\\ &= 2.5x^4 + 2x^3 - 2x^2 + 10x + c\end{aligned}$$

Note that integrating the constant, 10, gives $10x$.

Integration can be viewed as a summation process; for example, if you sum all of the marginal costs (the cost of producing one more) up to some point, then you will obtain total cost. This idea can also be used to find a sum of areas bounded by a curve between two points (known as definite integration), provided the function is positive. For example,

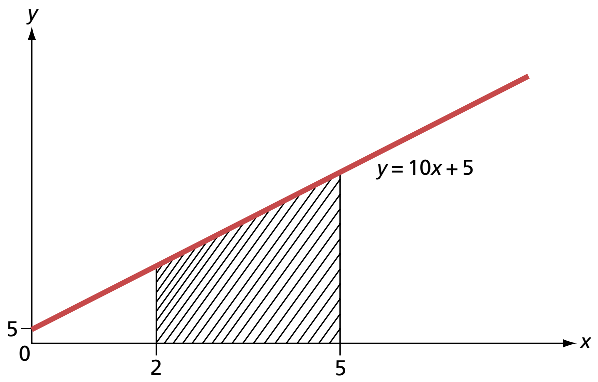
$$\int_2^5 (10x + 5)\,\mathrm{d}x$$

means find the integral of $10x + 5$, i.e. find the sum of all products $(10x + 5)\,\mathrm{d}x$, between the values $x = 2$ and $x = 5$.

Integrating gives the indefinite integral $5x^2 + 5x + c$, but the definite integral is usually written as

$$\left[5x^2 + 5x\right]_2^5$$

Figure 26.9



omitting the constant, because in the evaluation (below) it cancels itself out.

Now we evaluate this at $x = 5$, and at $x = 2$ and subtract the second from the first:

$$\begin{aligned}&(5x^2 + 5x)_{(x=5)} - (5x^2 + 5x)_{(x=2)}\\&= (125 + 25) - (20 + 10)\\&= 150 - 30\\&= 120\end{aligned}$$

This problem is illustrated in Figure 26.9, the shaded area being the 120 found above.

Within economics, the process of integration will allow us to go from marginal functions to total functions. Thus

$$\int \mathrm{MR}\,\mathrm{d}x = \mathrm{TR} + c$$

$$\int \mathrm{MC}\,\mathrm{d}x = \mathrm{TC} + c'$$

But with TR, there is rarely any revenue if output is zero, so in general the constant will also be zero. With TC there is a cost to the firm even if no production takes place, and so c' will be non-zero and positive. It will represent fixed cost.

If the marginal cost function for a firm has been identified as:

$$\mathrm{MC} = 2x^2 - 8x + 10$$

and fixed costs are known to be 100, then the total cost function is given by:

$$\begin{aligned}\mathrm{TC} &= \int \mathrm{MC}\,\mathrm{d}x\\&= \int (2x^2 - 8x + 10)\,\mathrm{d}x\\&= \frac{2}{3}x^3 - 4x^2 + 10x + c\end{aligned}$$

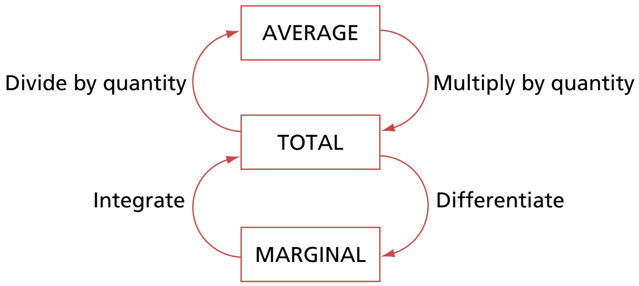
$\mathrm{TC} = 100$ when $x = 0$, therefore $c = 100$, and thus the total cost function is:

$$\mathrm{TC} = \frac{2}{3}x^3 - 4x^2 + 10x + 100$$

26.7 Economic summary

Figure 26.10 summarizes the application of calculus to simple economic models, usually of one firm or one market.

Figure 26.10



26.8 Functions of more than one variable

The simple economic models discussed so far embody functions of only one variable. As we move on to consider more complex economic and business situations, we find that there are *several* factors which will affect the outcome. Where the exact relationship is unknown, it must be estimated by statistical means (see Chapter 17), but for many examples in economics an exact function can be specified. For a duopoly (two sellers), the sales of one company's product may be a function of their own price, the price charged by the competitor and the respective amounts spent on advertising. In production theory, it is the *combination* of both capital and labour which determines the amount produced and not just one factor of production; while in welfare economics we consider the combination of goods which is available to the consumer or community. For all of these situations, while we are interested in the final outcome of particular decisions, we are also interested in the individual contribution of each factor, and also in ways of making optimal decisions on each factor in order to optimize the final outcome.

Consider a function such that y is determined by x and z; for example:

$$y = 10x^2 + 6xz + 15z^2$$

We can find the rate of change of y with respect to x if we *temporarily* hold the value of z constant when differentiating. (To distinguish this from differentiation where $y = f(x)$ we now use a new symbol $\delta y/\delta x$.)

Differentiating, we have:

$$\frac{\delta y}{\delta x} = 20x + 6z$$

Note that $10x^2$ was treated as before, giving $20x$, $6xz$ was treated as $(6z)x$, giving $6z$ and $15z^2$ was treated as a constant.

We can also find the rate of change of y with respect to z by temporarily holding x constant. Thus

$$\frac{\delta y}{\delta z} = 6x + 30z$$

The process described above is known as **partial differentiation** since we are finding a rate of change while part of the function is held constant. As with the functions of single variables, each of the functions we have obtained may be differentiated again, but note the number of outcomes:

$$\frac{\delta^2 y}{\delta x^2} = 20$$

i.e. holding z constant again and differentiating $20x + 6z$.

$$\frac{\delta^2 y}{\delta z^2} = 30$$

i.e. holding x constant again and differentiating $6x + 30z$.

$$\frac{\delta^2 y}{\delta x\,\delta z} = 6$$

i.e. holding x constant and differentiating $20x + 6z$.

$$\frac{\delta^2 y}{\delta z\,\delta x} = 6$$

i.e. holding z constant and differentiating $6x + 30z$.

Note, however, that

$$\frac{\delta^2 y}{\delta x\, \delta z} = \frac{\delta^2 y}{\delta z\, \delta x}$$

in general.

We may now use these results to find the maxima and minima for a particular function. For a **maximum**:

$$\frac{\delta y}{\delta x} = 0 \text{ and } \frac{\delta y}{\delta z} = 0$$

$$\frac{\delta^2 y}{\delta x^2} < 0 \text{ and } \frac{\delta^2 y}{\delta z^2} < 0$$

and

$$\left(\frac{\delta^2 y}{\delta x^2}\right)\left(\frac{\delta^2 y}{\delta z^2}\right) \geq \left(\frac{\delta^2 y}{\delta x\, \delta z}\right)^2$$

For a **minimum**:

$$\frac{\delta y}{\delta x} = 0 \text{ and } \frac{\delta y}{\delta z} = 0$$

$$\frac{\delta^2 y}{\delta x^2} > 0 \text{ and } \frac{\delta^2 y}{\delta z^2} > 0$$

and

$$\left(\frac{\delta^2 y}{\delta x^2}\right)\left(\frac{\delta^2 y}{\delta z^2}\right) \geq \left(\frac{\delta^2 y}{\delta x\, \delta z}\right)^2$$

EXAMPLES

1 If $y = 5x^2 + 4xz - 60x - 40z + 4z^2 + 1000$ then

$$\frac{\delta y}{\delta x} = 10x + 4z - 60 = 0 \quad \frac{\delta y}{\delta z} = 4x - 40 + 8z = 0$$

Rearranging gives two simultaneous equations:

$$10x + 4z = 60 \qquad 4x + 8z = 40$$

Multiplying the first by 2 gives

$$20x + 8z = 120$$

Subtracting the second gives

$$16x = +80$$

Therefore $x = 5$ and substituting gives $z = 2.5$

$$\frac{\delta^2 y}{\delta x^2} = 10, \quad \frac{\delta^2 y}{\delta z^2} = 8$$

both positive, and

$$\frac{\delta^2 y}{\delta x\, \delta z} = 4, \quad \left(\frac{\delta^2 y}{\delta x^2}\right)\left(\frac{\delta^2 y}{\delta z^2}\right) = 10 \times 8 = 80$$

which is greater than

$$\left(\frac{\delta^2 y}{\delta x\, \delta z}\right)^2 = 4^2 = 16$$

Thus the function has a minimum when $x = 5$ and $z = 2.5$ (the y value will be 800).

2 Consider a monopolist with two products, *A* and *B*, who wishes to maximize total profit. The two demand functions are:

$$P_A = 80 - q_A \qquad P_B = 50 - 2q_B$$

and the total cost function is

$$\text{TC} = 100 + 8q_A + 6q_B + 14q_A^2 + 4q_B^2 + 4q_Aq_B$$

The total revenue function will consist of two parts:

$$\text{TR}_A = 80q_A - q_A^2 \qquad \text{and} \qquad \text{TR}_B = 50q_B - 2q_B^2$$

thus the total profit function is

$$\begin{aligned} \pi &= \text{TR}_A + \text{TR}_B - \text{TC} \\ &= -100 + 72q_A + 44q_B - 15q_A^2 - 6q_B^2 - 4q_Aq_B \end{aligned}$$

$$\frac{\delta\pi}{\delta q_A} = 72 - 30q_A - 4q_B = 0$$

$$\frac{\delta\pi}{\delta q_B} = 44 - 12q_B - 4q_A = 0$$

Rearranging these equations gives two simultaneous equations:

$$\begin{aligned} 30q_A + 4q_b &= 72 \\ 4q_A + 12q_B &= 44 \end{aligned}$$

Multiplying the first by 3 gives

$$90q_A + 12q_B = 216$$

Then subtracting the second gives

$$86q_A = 172$$

Therefore $q_A = 2$ and substituting gives $q_B = 3$.

$$\frac{\delta^2\pi}{\delta q_A^2} = -30, \quad \frac{\delta^2\pi}{\delta q_B^2} = -12$$

both negative, and

$$\frac{\delta^2\pi}{\delta q_A \delta q_B} = -4$$

but $(-30)(-12) > (-4)^2$.

Thus maximum profit is where the quantity of *A* sold is two, and the quantity of *B* sold is three. By substitution into the profit function, we find that the level of profit will be 38. The prices charged for the two products are found from the demand functions; the price of *A* is 78 and the price of *B* is 44.

26.9 Maximization and minimization subject to constraints

Many economics problems have limitations upon the solution that may be obtained: production may be limited by the available supply of raw materials or by the capacity of the current factory; cost may have to be kept to a minimum subject to certain, predetermined, output levels; or a consumer's utility maximized subject to the budget. In each of these cases, if we simply maximize or minimize the function it is unlikely that this will also satisfy the constraint and thus we need to take the constraint into consideration during the optimization process. For simple functions is may be possible to substitute the constraint into the function, but the more general method is that of Lagrangian multipliers.

26.9.1 Substitution

For functions of two variables, we may be able to substitute the constraint into the function. Consider the following example:

$$z = 10 + 3x - 4x^2 + 10y - 2y^2 + xy$$

to be maximized subject to the constraint that $x = 2y$.

Substituting, we have:

$$\begin{aligned} z &= 10 + 6y - 16y^2 + 10y - 2y^2 + 2y^2 \\ &= 10 + 16y - 16y^2 \end{aligned}$$

$$\frac{dz}{dy} = 16 - 32y = 0 \text{ therefore } y = \tfrac{1}{2} \text{ and hence } x = 1$$

$$\frac{d^2z}{dy^2} = -32 \text{ therefore it is a maximum.}$$

EXERCISE

Find the values of *x* and *y* to maximize the function without the constraint. (Answer: $x = 22/31$; $y = 83/31$.)

26.9.2 Lagrangian multipliers

For more complex functions the method above will become tedious and thus we now place the constraint as part of the function we are to optimize. Again, we will work through an example. Suppose

$$z = 10x - 4x^2 + 20y - y^2 + 4xy$$

subject to $2x + 4y = 100$. The first step is to rearrange the constraint as follows:

$$100 - 2x - 4y = 0$$

We then multiply it by a new unknown value, λ (lambda), which is the Lagrangian multiplier:

$$\lambda(100 - 2x - 4y) = 0$$

and this function is added to the function z to give a new function z^*. Therefore

$$z^* = 10x - 4x^2 + 20y - y^2 + 4xy + \lambda(100 - 2x - 4y)$$

This will not change the value of z since, as we have just seen, the term added is equal to 0, and it can be shown that if we find the optimum point or points of the function z^* then these will be the optimum points for the original function z subject to the constraint.

Partially differentiating z^*, we have:

$$\frac{\delta z^*}{\delta x} = 10 - 8x + 4y - 2\lambda = 0$$
$$\frac{\delta z^*}{\delta y} = 20 - 2y + 4x - 4\lambda = 0$$

and

$$\frac{\delta z^*}{\delta \lambda} = 100 - 2x - 4y = 0$$

(thus satisfying the constraint).

As we have seen in Chapter 25, there are several ways of solving three simultaneous equations and these will not be repeated here: solving the three equations gives $x = 10$, $y = 20$ and $\lambda = 5$ (although strictly we do not need to know the value of λ).

Further differentiation shows that $x = 10$, $y = 20$ is a maximum for the function z subject to the constraint. (Note that although

$$\left(\frac{\delta^2 z}{\delta x^2}\right)\left(\frac{\delta^2 z}{\delta y^2}\right)$$

equals

$$\left(\frac{\delta^2 z}{\delta y\ \delta x}\right)^2$$

and is not 'greater than', there is sufficient evidence to show a maximum.)

EXAMPLE

Consider a Cobb–Douglas production function

$$Q = 10L^{1/2}K^{1/2}$$

subject to the constraint that $4L + 10K = 100$, where L is the amount of labour and K is the amount of capital.

Rearranging the constraint, we have

$$100 - 4L - 10K = 0 \qquad \text{and} \qquad \lambda(100 - 4L - 10K) = 0$$

We thus wish to maximize the function

$$Q^* = 10L^{1/2}K^{1/2} + \lambda(100 - 4L - 10K)$$

$$\frac{\delta Q^*}{\delta L} = 5L^{-1/2}K^{1/2} - 4\lambda = 0 \text{ so } \lambda = \frac{5K^{1/2}}{4L^{1/2}}$$

$$\frac{\delta Q^*}{\delta K} = 5L^{1/2}K^{-1/2} - 10\lambda = 0 \text{ so } \lambda = \frac{5L^{1/2}}{10K^{1/2}}$$

Thus

$$\frac{5K^{1/2}}{4L^{1/2}} = \frac{5L^{1/2}}{10K^{1/2}}$$

$$50K = 20L$$

$$L = 2.5K$$

But

$$\frac{\delta Q^*}{\delta \lambda} = 100 - 4L - 10K = 0$$

$$100 - 10K - 10K = 0$$

$$\text{so } K = 5 \text{ and hence } L = 12.5$$

Further partial differentiation shows this to be a maximum, and thus production (Q) is maximized subject to the constraint when $L = 12.5$ and $K = 5$. Note that the function Q itself, when unconstrained, has no maximum value.

EXERCISE

Check that the constraint is satisfied and find the value of Q (Answer: $Q = 79.06$.)

26.10 Conclusions

Calculus was developed during the seventeenth century from problems being explored by physical scientists, like Galileo, trying to describe the world. It arose from the need to give an adequate mathematical account of changes in motion. Although the more immediate applications of calculus were found in the physical sciences, calculus has played an important part in the development of other disciplines, particularly economics. Whenever we need to consider the effects of change, calculus provides a language and a set of techniques to manage the mathematical descriptions. If you need to take this subject area further, then visit the companion website where we look at equations with more than two variables and maximization or minimization subject to constraints.

26.11 Questions

Differentiate each of the following functions:

1 $y = 12$

2 $y = 10 + 2x$

3 $y = 3x + 7$

4 $y = 2x^5$

5 $y = 14x^3 - 3x^4$

6 $y = 17x^2 + 14x + 10$

7 $x = 7y + 2y^2 - 4y^3$

8 $p = 2q^2 - 10 + 4q^{-1}$

9 $y = e^x$

10 $r = 2s + 4s^2 + s^{1/2}$

11 $y = 20x^7 + 4x^6 - 3x^5 - 7x^4 + 4x^3 - 11x^2 + 2x + 57 - x^{-1} + 9x^{-2}$

12 $y = \frac{1}{x^2} - \frac{2}{x^3} - \frac{4}{3x^4}$

13 If average revenue for a company is given by:

$$\text{AR} = 100 - 2x \ (x \text{ is quantity})$$

find the marginal revenue function and the price when MR = 0.

Graph the AR, MR and TR functions.

14 A firm's total costs are given by the following function:

$$\text{TC} = \frac{1}{3}x^3 - 5x^2 + 30x \ (x \text{ is quantity})$$

Find the average cost and marginal cost functions. Graph the three functions.

15 Using the information in the previous two questions, find the quantity levels where:

(a) AR = AC; and
(b) MR = MC.

16 A firm's average cost function is given as:

$$\text{AC} = x^2 - 3x + 25 \ (x = \text{quantity})$$

and its total revenue function as:

$$\text{TR} = 60x - 2x^2$$

Find the quantity levels where:

(a) MC = MR; and
(b) AR = AC

17 Given the following demand function:

$$P = 40 - 2x$$

find the price elasticity of demand at quantity levels of:

(a) 8;
(b) 10;
(c) 12.

Construct a graph of the average revenue and marginal revenue functions. What may you determine from this graph and your previous calculations?

18 An oligopolist sells a product at £950 per unit of production, and estimates the average revenue function to consist of two linear segments.

Below ten units,	AR = 990 when output = 2
	AR = 965 when output = 7
Above ten units,	AR = 840 when output = 12
	AR = 565 when output = 17

(a) Find the equation of the average revenue function below ten units of output.
(b) Find the equation of the average revenue function above ten units of output.
(c) Find the equations of the *two* marginal revenue functions, assuming that total revenue is zero if output is zero.
(d) Sketch the average and marginal revenue functions.
(e) Find the price elasticity of demand for a price rise and for a price fall from the current position.
(f) Find the range of marginal cost figures that are consistent with profit maximization and the current output level. (NB. For profit maximization MC = MR.)

Find the maximum and/or minimum of each of the following functions, checking the second order conditions (i.e. d^2y/dx^2):

19 $y = x^2 - 10x + 25$

20 $y = -2x^2 + 40x + 1000$

21 $y = 3x^2 + 6x + 5$

22 $y = 2x + 1$

23 $y = 3x^2 - 10x + 4$

24 $y = \frac{1}{3}x^3 - 3x^2 + 5x + 10$

25 $a = -4b^2 + 2b + 10$

26 $y = 2x^3 - 12x^2 + 12x + 10$

27 $y = 25$

28 $y = 0.25x^4 - 2x^3 + 5.5x^2 - 6x + 40$

29 A company's profit function is given by:

$$\text{profit} = -2x^2 + 40x + 10 \text{ (where } x = \text{output)}$$

Determine the profit maximizing output level for the company, checking that the second order conditions are met.

30 A firm has the following total revenue and total cost functions:

$$\text{TR} = 100x - 2x^2$$

$$\text{TC} = \frac{1}{3}x^3 - 5x^2 + 30x \text{ (where } x = \text{output).}$$

Find the output level to maximize profits, and the level of profit achieved at this output.

31 A firm has an average revenue of

$$\text{AR} = 60 - 2x$$

and an average cost function of

$$\text{AC} = x^2 - 3x + 25 \text{ (where } x = \text{output)}$$

Find the output level for maximum profit, and the level of profit. (NB. Check the second order conditions.)

32 Given the following average cost function, show that the marginal cost function cuts the average cost function at the latter's minimum point.

$$\text{AC} = 2x^2 - 4x + 100$$

Use the rules of differentiation outlined in section 25.5 to find the first differentials of the following functions:

33 $y = (x^2 + 6)(2x + 5)$

34 $y = (3x^2 + 4x + 10)(2x^2 + 6x + 5)$

35 $y = (4x^3 + 6x^2)^3$

36 $y = e^{x^2/2}$

37 $y = (4x + 6x^2)/(x^3 + 6x)$

38 $y = 2x/(3x^4 + 10)$

Integrate each of the following functions:

39 $y = 4x$

40 $y = 3x^2 + 4x + 10$

41 $y = 2x^3 + 6x^2 + 10x^{-2}$

42 $y = 25$

43 $y = x^2 + x + 5$

44 $y = x + 10$ between limits of $x = 2$ and $x = 4$.

45 $y = 3x^2 + 4x + 5$ between limits of $x = 0$ and $x = 5$.

46 $y = -2x^2 + 40x + 10$ between limits of $x = 0$ and $x = 10$.

47 A firm's market demand function is

$$\text{AR} = 150 - 2x$$

and its marginal cost function is

$$\text{MC} = \frac{1}{3}x^2 - 5x + 30$$

(a) Find the level of output to maximize profits.
(b) Find the level of profit and the price at this point.
(c) Find the price elasticity of demand at this point.
(d) Sketch the total revenue, total cost and profit functions.
(e) If a tax of 10 per unit is imposed on the firm, what effect will this have on the production level for maximum profit?

48 A company is faced by the following marginal cost and marginal revenue functions:

$$\text{MC} = 16 - 2Q$$
$$\text{MR} = 40 - 16Q$$

It is also known that fixed costs are 8 when production is zero. Find:

(a) the total cost function,
(b) the total revenue function,
(c) the output to give maximum sales revenue,
(d) the output to give maximum profit,
(e) the range of output for which profit is at least 1; and
(f) sketch the total cost and total revenue functions.

Find the first partial derivatives for each of the following:

49 $y = 4xz$

50 $y = 2x^3 + 4x^2z + 2xz^2 - 3z^3$

51 $y = 7x + 3x^2z - x^3z^2 - 2xz + 5xz^2 - 7xz^3 - 10$

52 $y = 10x^4 - 15z^3$

53 $q = 2p_1^2 - 3p_1 + 4p_1p_2 - 5p_2 + p_2^2$

54 $r = 2t(s^2 + s + 5) - 3s(2t^2 - 4t + 7)$

Find the maxima or minima of the following functions:

55 $y = 2x^2 - 2xz + 2.5z^2 - 2x - 11z + 20$

56 $y = 3x^2 + 3xz + 3z^2 - 21x - 33z + 100$

57 $y = -2x^2 - 2xz - 4z^2 + 40x + 90z - 150$

58 $z = 19x - 4x^2 + 16y - 2xy - 4y^2$

59 $y = x^2 + xz + z^2$

60 $y = 100 - 4x$

61 If a firm's total costs are related to its workforce and capital equipment by the function

$$\text{TC} = 10L^2 + 10K^2 - 25L - 50K - 5LK + 2000$$

where L = thousands of employees and K = thousands of pounds invested in capital equipment, find the combination of labour and capital to give minimum total cost. Find this cost and show that it is a minimum.

62 A monopolist has two products, X and Y, which have the following demand functions:

$$P_X = 26.2 - X$$
$$P_Y = 24 - 2Y$$

The total costs of the monopolist are given by:

$$\text{TC} = 5X^2 + XY + 3Y^2$$

Determine the amounts of X and Y the monopolist should produce to maximize profit, and the amount of profit produced.

63 A company is able to sell two products, X and Y, which have demand functions:

$$P_X = 52 - 2X$$
$$P_Y = 20 - 3Y$$

and has a total cost function

$$\text{TC} = 10 + 3X^2 + 2Y^2 + 2XY$$

(a) Determine the profit maximizing levels of output for X and Y and the level of this profit.
(c) If product Y were not produced, determine the production level of X to maximize profit and the level of this profit.
(c) If product X were not produced, determine the production level of Y to maximize profit and the level of this profit.

Maximize or minimize the following functions subject to their constraints, using either substitution or the method of Lagrangian multipliers.

64 $y = 10 + 20x + 6z - 4x^2 - 2z^2$ subject to $x = 3z$.

65 $y = 4x^2 - 6x + 7z + 3z^2 + 2xz$ subject to $x = 5z$.

66 $z = 10x - x^2 + 14y - 2y^2 + xy$ subject to $2x + 3y = 3100$.

67 $z = 6x - 3x^2 + 40y - 8y^2 + 5xy$ subject to $4x - 5y = 30$.

68 $z = 2xy$ subject to $x + y = 1$.

69 Maximize the production function

$$Q = 4L^{1/2}K^{1/2}$$

subject to the constraint that $3L + 5K = 200$, finding the values for L and K.

70 Find the values of L and K to maximize the production function

$$Q = 8L^{1/4}K^{3/4}$$

subject to the constraint $L + 8K = 1000$.

71 A firm's profit is given by the function

$$\pi = 600 - 3x^2 - 4x + 2xy - 5y^2 + 48y$$

where π denotes profit, x output and y advertising expenditure.

(a) Find the profit maximizing values of x and y and hence determine the maximum profit. Confirm that second order conditions are satisfied.
(b) If now the firm is subject to a budget constraint $2x + y = 5$ determine the new values of x and y which maximize profit.
(c) Without further calculation determine the effect of a constraint $2x + y = 8$.